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STUDY OF CERTAIN NEW TENSORS IN A FINSLER SPACE OF THREE-DIMENSIONS

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ABSTRACT

In one of his earlier papers in (1990), the author [7] has defined and studied several new tensors of second and third order. The author, further in Ref. [9] has defined and studied several properties of asymmetric third order new tensor D_{ijk}, which is similar to the Cartan's torsion tensor [1], C_{ijk} of a Finsler space of three dimensions. This tensor, however, satisfies $D_{ijk}\, l^i = 0$, $D_{ijk}\, g^{jk} = D_i = D\, n_i$. Based on this tensor, author has defined several other tensors including $Q_{ijk} = D_{ijk/0}$, similar to P_{ijk} and $D^{`}_{ijkh}$ similar to third curvature tensor S_{ijkh}. The purpose of the present paper is to define and study a tensor U_{ijkkh}, similar to second curvature tensor P_{ijkh} and V_{ijkh} similar to a very important tensor T_{ijkh}, which was introduced independently by Kawaguchi [3] and Matsumoto [5] in (1972).

KEYWORDS: *Finsler Space F^3, Tensors D_{ijk}, Q_{ijk}, U_{ijkh} and V_{ijkh}*

INTRODUCTION

Let F^3 be a three-dimensional Finsler space with the Moor's frame (l_i, m_i, n_i). Corresponding to this frame, the metric tensor, angular metric tensor and (h) hv-torsion tensors are given by Matsumoto [6] and Rund [10] as

$$g_{ij} = l_i l_j + m_i\, m_j + n_i\, n_j,\; h_{ij} = m_i\, m_j + n_i n_j, \tag{1.1}$$

and

$$C_{ijk} = C_{(1)}\, m_i\, m_j m_k + C_{(2)} n_i\, n_j n_k + \Sigma_{(ijk)} \{C_{(3)}\, m_i\, n_j n_k - C_{(2)}\, m_i\, m_j n_k\} \tag{1.2}$$

In a three-dimensional Finsler space F^3, h- and v-covariant derivatives of the unit vectors l^i, m^i and n^i are defined as [6]

$$l^i_{/j} = 0,\; m^i_{/j} = n^i h_j,\; n^i_{/j} = -\, m^i h_j \tag{1.3}$$

and

$$l^i_{//j} = L^{-1} h^i_j,\; m^i_{//j} = L^{-1}(-l^i\, m_j + n^i v_j),\; n^i_{//j} = -L^{-1}(l^i\, n_j + m^i v_j) \tag{1.4}$$

where $v_j = v_{2)3\gamma}\, e_{\gamma)j}$ and $h_j = H_{2)3\,\gamma} e_{\,\gamma)j}$.

In, general for a tensor field K^h_m, we have

$$K^h_{m\,/\,r} = \partial_r K^h_m - N^j_r \Delta_j K^h_m + K^k_m F_k{}^h{}_r - K^h_k F_m{}^k{}_r \tag{1.5}$$

and

$K^h_{m//r} = \Delta_r K^h_m + K^j_m C_j{}^h{}_r - K^h_j C_m{}^j{}_r$ (1.6)

where $\partial_r = \partial/\partial x^r$ and $\Delta_r = \partial/\partial y^r$.

The second and third curvature tensors in the sense of E. Cartan [1] are given by

$P_{ijkh} = \varsigma_{(i,j)} \{A_{jkh/i} + A_{ikr}P^r_{jh}\}$ (1.7)

$S_{ijkh} = \varsigma_{(h,k)}\{A_{ihr}A^r_{jk}\}$ (1.8)

such that

$\varsigma_{(h,k)}\{P_{ijkh}\} = - S_{ijkh/0}$(1.9)

where $\varsigma_{(h,k)}\{\}$ means interchange of indices h and k and subtraction.

The tensor D_{ijk} was introduced and defined by Rastogi [9], such that it satisfies $D_{ijk}\ l^i = 0$, $D_{ijk}\, g^{jk} = D_i = D\, n_i$ and is given as

$D_{ijk} = D_{(1)}\, m_i\, m_j m_k + D_{(2)} n_i\, n_j n_k + \sum_{(i,j,k)}\{D_{(3)}\, m_i\, m_j n_k$

$+ D_{(4)}\, m_i\, n_j n_k\}$ (1.10)

where $D_{(1)}$, $D_{(2)}$, $D_{(3)}$ and $D_{(4)}$ are scalars satisfying.

$D_{(2)} + D_{(3)} = D,\ D_{(1)} + D_{(4)} = 0$ (1.11)

Tensor U_{ijkh}

Corresponding to second curvature tensor of Cartan [1], P_{ijkh}, we here define the curvature tensor U_{ijkh} as follows:

$U_{ijkh} = \varsigma_{(i,j)} \{D_{jkh/i} + D_{ikr}Q^r_{jh}\}$ (2.1)

From equation (1.10), we can get

$D_{jkh/i} = m_j m_k m_h(D_{(1)/i} - 3\, D_{(3)}\, h_i) + n_j n_k n_h(D_{(2)/i} - 3\, D_{(1)}\, h_i) + \sum_{(j,k,h)}[m_j m_k n_h\ \{D_{(3)/i} + (D_{(1)} - 2\, D_{(4)})\, h_i\} - m_j n_k n_h\ \{D_{(1)/i} + (D_{(2)} - 2\, D_{(3)})\, h_i\}]$ (2.2)

whereas $Q_{ijk} = D_{ijk/0}$, can be given as

$Q_{ijk} = \{D_{(1)/0} - 3\, D_{(3)}\, h_0\}\, m_i\, m_j m_k + \{(D_{(2)/0} - 3\, D_{(1)}\, h_0)\, n_i\, n_j n_k\} + \sum_{(I,j,k)}[\{D_{(3)/0} + 3\, D_{(1)}\, h_0\}\, m_i\, m_j n_k - \{D_{(1)/0} + (D_{(2)} - 2\, D_{(3)})\, h_0\}\, m_i\, n_j n_k]$ (2.3)

From equation (2.1), by virtue of (2.2) and (2.3), we can obtain

$U_{ijkh} = {}^1A_{ij} m_k m_h + {}^2A_{ij} m_k n_h + {}^3A_{ij} m_h n_k + {}^4A_{ij} n_h n_k$, (2.4)

where, we have assumed

${}^1A_i = D_{(1)/I} - 3\, D_{(3)}\, h_i,\ {}^2A_i = D_{(1)/I} + (D_{(2)} - 2\, D_{(3)})h_i$, (2.5) a

${}^3A_i = D_{(2)/I} - 3\, D_{(1)}\, h_i,\ {}^4A_i = D_{(3)/I} + 3\, D_{(1)}\, h_i$ (2.5) b

and

${}^1A_0 = {}^1A_i\, l^i,\ {}^2A_0 = {}^2A_i\, l^i,\ {}^3A_0 = {}^3A_i\, l^i,\ {}^4A_0 = {}^4A_i\, l^i$, (2.5) c such that

$$^1A_{ij} = Ç_{(I,j)} [^1A_i m_j + {}^4A_i n_j + \{D_{(3)} (^1A_0 - {}^2A_0) - (D_{(1)} - D_{(4)}) {}^4A_0\} m_j n_i] \quad (2.6)a$$

$$^2A_{ij} = Ç_{(I,j)} [^4A_i m_j - {}^2A_i n_j + \{D_{(3)} (^4A_0 - {}^3A_0) + (D_{(1)} - D_{(4)}) {}^2A_0\} m_j n_i] \quad (2.6)b$$

$$^3A_{ij} = Ç_{(I,j)} [^4A_i m_j - {}^2A_i n_j + \{D_{(4)} (^1A_0 - {}^2A_0) + (D_{(2)} - D_{(3)}) {}^4A_0\} m_j n_i] \quad (2.6)c$$

$$^4A_{ij} = Ç_{(I,j)} [^2A_j m_i - {}^3A_j n_i + \{D_{(4)} (^4A_0 - {}^3A_0) - (D_{(2)} - D_{(3)}) {}^2A_0\} m_j n_i] \quad (2.6)d$$

From equation (2.4), by virtue of equations (2.5) and (2.6), we can obtain

$$U_{ijkh} l^i = Q_{jkh}, U_{ijkh} g^{ij} = 0, \quad (2.7) a$$

$$U^*_{ij} = U_{ijkh} g^{kh} = {}^1A_{ij} + {}^4A_{ij}, U^*_{ij} + U^*_{ji} = 0, \quad (2.7)b$$

$$U_{ijkh} - U_{ijhk} = \{^4A_0 (D_{(2)} - 2D_{(3)}) + {}^3A_0 D_{(3)} - D_{(1)} (^1A_0 + {}^2A_0)\}$$

$$(m_k n_h - m_h n_k).(m_i n_j - m_j n_i). \quad (2.7)c$$

In general, $m_k n_h \neq m_h n_k$, therefore from equation (2.7) c we can obtain

Theorem 2.1

In a three-dimensional Finsler space F^3, the necessary and sufficient condition for the tensor U_{ijkh} to be symmetric in k and h is given by $^4A_0 (D_{(2)} - 2D_{(3)}) + {}^3A_0 D_{(3)} - (^1A_0 + {}^2A_0) D_{(1)} = 0$.

In F^3, it is known that Matsumoto [6]

$$h_{ik} h_{jh} - h_{ih} h_{jk} = (m_i n_j - m_j n_i) (m_k n_h - m_h n_k),$$

Therefore, equation (2.7) c, can also be expressed as

$$Ç_{(k,h)}[U_{ijkh} - \{^4A_0 (D_{(2)} - 2D_{(3)}) + {}^3A_0 D_{(3)} - (^1A_0 + {}^2A_0) D_{(1)}\} h_{ik} h_{jh}] = 0 \quad (2.8)$$

It is also known that the tensor D'_{ijkh} can be expressed as [9]:

$$D'_{ijkh} = (2 D_{(1)}^2 - D_{(2)} D_{(3)} + D_{(3)}^2) (h_{ik} h_{jh} - h_{ih} h_{jk}), \quad (2.9)a$$

Such that

$$D'_{ik} = D'_{ijkh} g^{jh} = (2 D_{(1)}^2 - D_{(2)} D_{(3)} + D_{(3)}^2) h_{ik} \quad (2.9)b$$

Therefore, from equations (2.8) and (2.9) a, we can obtain

$$Ç_{(k,h)}[U_{ijkh} - \{^4A_0 (D_{(2)} - 2D_{(3)}) + {}^3A_0 D_{(3)} - (^1A_0 + {}^2A_0) D_{(1)}\}.$$

$$D'_{ijkh} (2 D_{(1)}^2 - D_{(2)} D_{(3)} + D_{(3)}^2)^{-1}] = 0 \quad (2.10)$$

Hence:

Theorem 2.2

In a three-dimensional Finsler space F^3, curvature tensors U_{ijkh} and D'_{ijkh} are related by equation (2.10).

Also, from equations (2.8) and (2.9), one can obtain after simplification

$Ç_{(k,h)}\{U_{ijkh}\} = - D'_{ijkh/0} + 2\, D_{(1)}\,{}^{2}A_0(h_{ik}h_{jh} - h_{ih}h_{jk})$ (2.11)

Hence:

Theorem 2.3

In a three-dimensional Finsler space F^3, curvature tensor U_{ijkh} and $D'_{ijkh/0}$ are related by equation (2.11).

From equation (2.4), one can also imagine that the tensor U_{ijkh} can be expressed as the difference of product of two tensors in the following form:

$U_{ijkh} = S_{ik}T_{jh} - B_{jk}T_{ih}$, (2.12)

where we have assumed

$S_{ik} = a_i m_k + b_i n_k$ (2.13)a

and

$T_{jh} = d_j m_h + e_j n_h$ (2.13)b

such that the vectors a_i, b_i, d_j and e_j satisfy following relations:

${}^{1}A_{ij} = a_i d_j - a_j d_i,\ {}^{2}A_{ij} = a_i e_j - a_j e_i$, (2.14)a

${}^{3}A_{ij} = b_i d_j - b_j d_i,\ {}^{4}A_{ij} = b_i e_j - b_j e_i$ (2.14)b

Using equations (2.6) a, b, c, d in equations (2.13) a, b, we can obtain

$a_i d_j - a_j d_i = {}^{1}A_i m_j + {}^{4}A_i n_j - {}^{1}A_j m_i - {}^{4}A_j n_i + A(m_j n_i - m_i n_j)$, (2.15) a

$a_i e_j - a_j e_i = {}^{4}A_i m_j - {}^{2}A_i n_j - {}^{4}A_j m_i + {}^{2}A_j n_i + B(m_j n_i - m_i n_j)$, (2.15)b

$b_i d_j - b_j d_i = {}^{4}A_i m_j - {}^{2}A_i n_j - {}^{4}A_j m_i + {}^{2}A_j n_i + C(m_j n_i - m_i n_j)$, (2.15) c

$b_i e_j - b_j e_i = {}^{2}A_j m_i - {}^{3}A_j n_i - {}^{2}A_i m_j + {}^{3}A_i n_j + E(m_j n_i - m_i n_j)$ (2.15) d

where

$A = D_{(3)}({}^{1}A_0 - {}^{2}A_0) - (D_{(1)} - D_{(4)})\,{}^{4}A_0,\ B = D_{(3)}({}^{4}A_0 - {}^{3}A_0) + (D_{(1)} - D_{(4)})\,{}^{2}A_0$,

$C = D_{(4)}({}^{1}A_0 - {}^{2}A_0) + (D_{(2)} - D_{(3)})\,{}^{4}A_0,\ E = D_{(4)}({}^{4}A_0 - {}^{3}A_0) - (D_{(2)} - D_{(3)})\,{}^{2}A_0$.

Using $a_i m^i = a'$, $a_i n^i = a^*$, $b_i m^i = b'$, $b_i n^i = b^*$ etc., in (2.15) a, we can get

$a' d_j - a_j d' = ({}^{1}A_i m^i) m_j + ({}^{4}A_i m^i) n_j - {}^{1}A_j - A n_j$ (2.16)a

$a^* d_j - a_j d^* = ({}^{1}A_i n^i) m_j + ({}^{4}A_i n^i) n_j - {}^{4}A_j + A m_j$ (2.16)b

By solving equations (2.16) a and (2.16) b, we can obtain values of a_j and d_j in the following form:

$a_j = (a' d^* - a^* d')^{-1} [m_j \{({}^{1}A_i m^i) a^* - ({}^{1}A_i n^i - A) a'\} + n_j \{({}^{4}A_i m^i - A) a^* - ({}^{4}A_i n^i) a'\} - ({}^{1}A_j a^* - {}^{4}A_j a')]$ (2.17)a

$d_j = (a' d^* - a^* d')^{-1} [m_j \{({}^{1}A_i m^i) d^* - ({}^{1}A_i n^i + A) d'\} + n_j \{({}^{4}A_i m^i - A) d^* - ({}^{4}A_i n^i) d'\} + ({}^{4}A_j d' - {}^{1}A_j d^*)]$ (2.17)b

Similarly, from equations (2.15) b, c, d, we can obtain

$a_j = (e^* a' - a^* e')^{-1}[\, m_j \{({}^4A_i m^i) a^* - ({}^4A_i n^i + B) a'\} + n_j \{({}^2A_i n^i) a' - ({}^2A_i m^i + B) a^*\} - ({}^4A_j a^* + {}^2A_j a')]$, (2.17)c

$e_j = (e^* a' - a^* e')^{-1}[\, m_j \{({}^4A_i m^i) e^* - ({}^4A_i n^i + B) e'\} + n_j \{({}^2A_i n^i) e' - ({}^2A_i m^i + B) e^*\} - ({}^2A_j e' + {}^4A_j e^*)]$, (2.17)d

$b_j = (b' d^* - b^* d')^{-1}[\, m_j \{({}^4A_i m^i) b^* - ({}^4A_i n^i + C) b'\} + n_j \{({}^2A_i n^i) b' - ({}^2A_i m^i + C) b^*\} - ({}^2A_j b' + {}^4A_j b^*)]$, (2.17)e

$d_j = (b' d^* - b^* d')^{-1}[\, m_j \{(({}^4A_i m^i) d^* - ({}^4A_i n^i + C) d'\} + n_j \{({}^2A_i n^i) d' - ({}^2A_i m^i + C) d^*\} - ({}^2A_j d' + {}^4A_j d^*)]$,(2.17)f

$b_j = (b' e^* - b^* e')^{-1}[\, m_j \{({}^2A_i n^i - E) b' - ({}^2A_i m^i) b^*\} + n_j \{({}^3A_i m^i - E) b^* - ({}^3A_i n^i) b'\} + ({}^3A_j b' + {}^2A_j b^*)]$, (2.17)g

$e_j = (b' e^* - b^* e')^{-1}[\, m_j \{({}^2A_i n^i - E) e' - ({}^2A_i m^i) e^*\} + n_j \{({}^3A_i m^i - E) e^* - ({}^3A_i n^i) e'\} + ({}^2A_j e^* + {}^3A_j e')]$ (2.17)h

Remark

In equations (2.17), we have obtained two values each for vectors a_i, b_i, d_i and e_i. By equating these values, we get

$({}^1A_i n^i - {}^4A_i m^i + A)(e^*a' - a^*e') = ({}^2A_i m^i + {}^4A_i n^i + B)(a'd^* - a^*d')$ 2.18)a

$({}^2A_i n^i + {}^3A_i m^i - E)(b'd^* - b^*d') = ({}^2A_i m^i + {}^4A_i n^i + C)(b'e^* - b^*e')$ (2.18)b

$({}^4A_i m^i - {}^1A_i n^i - A)(b'd^* - b^*d') = ({}^2A_i m^i + {}^4A_i n^i + C)(a'd^* - a^*d')$ (2.18)c

$({}^2A_i m^i + {}^4A_i n^i + B)(b'e^* - b^*e') = ({}^2A_i n^i + {}^3A_i m^i - E)(a'e^* - a^*e')$ (2.18)d

Hence:

Theorem 2.4

In a three-dimensional Finsler space F^3, if the tensor U_{ijkh} is expressed as in (2.12), the coefficients ${}^1A_i$, ${}^2A_i$, ${}^3A_i$, ${}^4A_i$ and constants a*, b*, d*, e*, a', b', d' and e' satisfy equations (2.18) a, b, c, d.

U-Ricci Tensors

Let us assume that we define two tensors with the help of equation (2.4) as follows:

$U_{ijkh} g^{jk} = U_{ih}^{(1)} = {}^1A_{ij} m^j m_h + {}^2A_{ij} m^j n_h + {}^3A_{ij} n^j m_h + {}^4A_{ij} n^j n_h$ (3.1)

and

$U_{ijkh} g^{jh} = U_{ik}^{(2)} = {}^1A_{ij} m^j m_k + {}^2A_{ij} n^j m_k + {}^3A_{ij} m^j n_k + {}^4A_{ij} n^j n_k$ (3.2)

The tensors $U_{jh}^{(1)}$ and $U_{jh}^{(2)}$ are tensors similar to Ricci-tensors for curvature tensor P_{ijkh}, defined and studied by Shimada [11].

Using equations (2.6) a, b, c, d and ${}^1A_{ij} m^j = {}^1A^*_I$, ${}^2A_{ij} m^j = {}^2A^*_i$, ${}^3A_{ij} m^j = {}^3A^*_I$, ${}^4A_{ij} m^j = {}^4A^*_I$, ${}^1A_{ij} n^j = {}^1A'_I$, ${}^2A_{ij} n^j = {}^2A'_I$, ${}^3A_{ij} n^j = {}^3A'_I$ and ${}^4A_{ij} n^j = {}^4A'_I$, in equations (3.1) and (3.2), we get

$U_{ih}^{(1)} = ({}^1A^*_i + {}^3A'_I) m_h + ({}^2A^*_i + {}^4A'_i) n_h$ (3.3)

and

$U_{ik}^{(2)} = ({}^1A^*_i + {}^2A'_i) m_k + ({}^3A^*_i + {}^4A'_i) n_k$ (3.4)

where

${}^1A^*_I = {}^1A_i - m_i ({}^1A_j m^j) - n_i ({}^4A_j m^j) + \{D_{(3)}({}^1A_0 - {}^2A_0) - 2\,{}^4A_0 D_{(1)}\} n_i$, (3.5)a

$$^{1}A'_{I} = {}^{4}A_{i} - m_{i}({}^{1}A_{j}\, n^{j}) - n_{i}({}^{4}A_{j}\, n^{j}) - \{D_{(3)}({}^{1}A_{0} - {}^{2}A_{0}) - 2\,{}^{4}A_{0}\, D_{(1)}\}\, m_{i}, \quad (3.5)b$$

$$^{2}A^{*}_{I} = {}^{4}A_{i} - m_{i}({}^{4}A_{j}\, m^{j}) + n_{I}({}^{2}A_{j}\, m^{j}) + \{D_{(3)}({}^{4}A_{0} - {}^{3}A_{0}) + 2\,{}^{2}A_{0}\, D_{(1)}\}\, n_{i}, \quad (3.5)c$$

$$^{2}A'_{I} = -{}^{2}A_{i} - m_{i}({}^{4}A_{j}\, n^{j}) + n_{i}({}^{2}A_{j}\, n^{j}) - \{D_{(3)}({}^{4}A_{0} - {}^{3}A_{0}) + 2\,{}^{2}A_{0}\, D_{(1)}\}\, m_{i}, \quad (3.5)d$$

$$^{3}A^{*}_{I} = {}^{4}A_{i} - m_{i}({}^{4}A_{j}\, m^{j}) + n_{i}({}^{2}A_{j}\, m^{j}) + \{D_{(1)}({}^{2}A_{0} - {}^{1}A_{0}) + {}^{4}A_{0}(D_{(2)} - D_{(3)})\}n_{i}, \quad (3.5)e$$

$$^{3}A'_{I} = -{}^{2}A_{i} - m_{i}({}^{4}A_{j}n^{j}) + n_{i}({}^{2}A_{j}\, n^{j}) - \{D_{(1)}({}^{2}A_{0} - {}^{1}A_{0}) + {}^{4}A_{0}(D_{(2)} - D_{(3)})\}\, m_{i}, \quad (3.5)f$$

$$^{4}A^{*}_{I} = -{}^{2}A_{i} + m_{i}({}^{2}A_{j}\, m^{j}) - n_{i}({}^{3}A_{j}\, m^{j}) + \{D_{(1)}({}^{3}A_{0} - {}^{4}A_{0}) + {}^{2}A_{0}(D_{(2)} - D_{(3)})\}n_{i}, \quad (3.5)g$$

$$^{4}A'_{I} = {}^{3}A_{i} + m_{i}({}^{2}A_{j}\, n^{j}) - n_{i}({}^{3}A_{j}\, n^{j}) + \{D_{(1)}({}^{3}A_{0} - {}^{4}A_{0}) + {}^{2}A_{0}(D_{(2)} - D_{(3)})\}\, m_{i} \quad (3.5)h$$

Using equations (3.5) a, b, c, d, e, f, g, h in (equations (3.3) and (3.4), we can establish

$$Ç_{(I,h)}[U_{ih}^{(1)} - m_{h}({}^{1}A_{i} - {}^{2}A_{i}) - n_{h}({}^{3}A_{i} + {}^{4}A_{i}) - m_{i}n_{h}\{D_{(1)}({}^{3}A_{0} + {}^{4}A_{0}) + D_{(2)}{}^{2}A_{0} - D_{(3)}{}^{3}A_{0}\}] = 0 \quad (3.6)$$

and

$$Ç_{(I,h)}[U_{ih}^{(2)} - m_{h}({}^{1}A_{i} - {}^{2}A_{i}) - n_{h}({}^{3}A_{i} + {}^{4}A_{i}) - m_{i}n_{h}\{D_{(1)}\,({}^{3}A_{0} + {}^{4}A_{0}) + D_{(2)}{}^{2}A_{0} - D_{(3)}{}^{1}A_{0}\}] = 0 \quad (3.7)$$

These tensors also satisfy

$$U_{i0}^{(1)} = 0,\ U_{i0}^{(2)} = 0,\ U_{0h}^{(1)} = ({}^{1}A_{0} - {}^{2}A_{0})\, m_{h} + ({}^{4}A_{0} + {}^{3}A_{0})\, n_{h}, \quad (3.8)$$

$$U_{0h}^{(2)} = ({}^{1}A_{0} - {}^{2}A_{0})\, m_{h} + ({}^{3}A_{0} + {}^{4}A_{0})\, n_{h} = U_{0h}^{(1)} \quad (3.9)$$

Hence:

Theorem 3.1

In a three-dimensional Finsler space F^3, the Ricci-tensors based on curvature tensor U_{ijkh} and defined by equations (3.3) and (3.4) satisfy equations (3.6), (3.7), (3.8) and (3.9).

Tensor V_{ijkh}

We now define a tensor V_{ijkh}, based on third order tensor D_{ijk} as follows:

Definition 4.1

In a three-dimensional Finsler space F^3, based on third order symmetric tensor D_{ijk}, in analogy to tensor T_{ijkh}, we here define the tensor V_{ijkh} as follows:

$$V_{ijkh} = L\, D_{ijk//h} + l_{h}\, D_{ijk} + l_{k}D_{ijh} + l_{j}D_{ikh} + l_{i}D_{jkh} \quad (4.1)$$

Substituting value of D_{ijk} from equation (1.10), in (4.1), after some simplification by virtue of equation (1.4), we get

$$V_{ijkh} = \alpha_{h}\, m_{i}\, m_{j}m_{k} + \textstyle\sum_{(I,j,k)} (\beta_{h}\, m_{i}\, m_{j}n_{k} - \gamma_{h}\, m_{i}\, n_{j}n_{k}) + \delta_{h}n_{i}\, n_{j}n_{k} \quad (4.2)$$

where

$$\alpha_{h} = L\, D_{(1)//h} + l_{h}\, D_{(1)} - 3\, v_{h}\, D_{(3)}, \quad (4.3)a$$

$$\beta_{h} = L\, D_{(3)//h} + l_{h}\, D_{(3)} + 3\, v_{h}\, D_{(1)}, \quad (4.3)b$$

$$\gamma_{h} = -\,\{L\, D_{(1)//h} + l_{h}\, D_{(1)} + v_{h}(3\, D_{(2)} - 2D)\}, \quad (4.3)c$$

$\delta_h = L\, D_{(2)//h} + l_h\, D_{(2)} - 3\, v_h\, D_{(1)}$ (4.3) d

From equation (2.2), we can get $V_{ijkh} = V_{jikh}$, i.e., V_{ijkh} is symmetric in first two indicesi and j. Also

$V_{ijkh} – V_{ijhk} = m_i\, m_j\, (\alpha_h m_k – \alpha_k m_h + \beta_h n_k – \beta_k n_h) + n_i\, n_j\, (\gamma_h m_k - \gamma_k m_h)$

$+ (m_i\, n_j + m_j n_i)\, (\beta_h m_k – \beta_k m_h + \gamma_h n_k – \gamma_k n_h),$ (4.4)

$(V_{ijkh} - V_{ijhk})\, g^{kh} = 0,$ (4.5) a

and

$(V_{ijkh} – V_{ijhk})\, g^{ij} = D\, (v_k m_h – v_h m_k) + (\beta_h n_k - \beta_k n_h).$ (4.5)b

Hence:

Theorem 4.1

In a three-dimensional Finsler space F^3, tensor V_{ijkh} is symmetric in i and j and satisfies equations (4.4), (4.5) a and (4.5) b.

In case we assume that V_{ijkh} is also symmetric in k and h, equation (4.4) on simplification will give

$(\alpha_h + \gamma_h)\, m_k – (\alpha_k + \gamma_k)\, m_h + \beta_h n_k – \beta_k n_h = 0$, which by virtue of equations (4.3) a, b, c leads to

$\beta_h m^h + D\, v_h n^h = 0$ (4.6)a

or alternatively

$\beta_h m^h + D\, v_{2)33} = 0.$ (4.6)b

Hence:

Theorem 4.2

In a three-dimensional Finsler space F^3, if the tensor V_{ijkh} is symmetric in k and h, equation (4.6) b, is satisfied.

From equations (4.3) a, b, c, d we can obtain

$\alpha_0 = \alpha_h l^h = L\, D_{(1)//0} + D_{(1)},\ \beta_0 = \beta_h l^h = L\, D_{(3)//0} + D_{(3)},$

$\gamma_0 = \gamma_h l^h = - L\, D_{(1)//0} – D_{(1)},\ \delta_0 = \delta_h l^h = L\, D_{(2)//0} + D_{(2)}$

$\alpha_h m^h = L\, D_{(1)//h} m^h – 3\, D_{(3)}\, v_{2)32},\ \beta_h m^h = L\, D_{(3)//h} m^h + 3\, D_{(1)}\, v_{2)32}$

$\gamma_h m^h = - L\, D_{(1)//h} m^h - (3\, D_{(2)} – 2\, D) v_{2)32},\ \delta_h m^h = L\, D_{(2)//h} m^h – 3\, D_{(1)}\, v_{2)32},$

$\alpha_h n^h = L\, D_{(1)//h} n^h – 3\, D_{(3)}\, v_{2)33},\ \beta_h n^h = L\, D_{(3)//h} n^h + 3\, D_{(1)}\, v_{2)33},$

$\gamma_h n^h = - L\, D_{(1)//h} n^h – (3\, D_{(2)} - 2D) v_{2)33},$

$\delta_h n^h = L\, D_{(2)//h} n^h – 3\, D_{(1)}\, v_{2)33},$ (4.7)

which show

Theorem 4.3

In a three-dimensional Finsler space

- $\alpha_0 + \gamma_0 = 0$, ii)$\beta_0 + \delta_0 = L\, D_{//0} + D,$

- $(\alpha_h + \gamma_h)\, m^h + D\, v_{2)32} = 0$, iv) $(\alpha_h + \gamma_h)\, n^h + D\, v_{2)33} = 0$,
- $(\beta_h + \delta_h)\, m^h = L\, D_{//h} m^h$ and vi) $(\beta_h + \delta_h)\, n^h = L\, D_{//h} n^h$.

From equation (3.2), we can observe that

$V_{ijkh} g^{ij} = (\alpha_h - \gamma_h)\, m_k + (\beta_h + \delta_h)\, n_k$

and

$V_{ijkh} g^{kh} = (\alpha_h m^h + \beta_h n^h)\, m_i\, m_j + (\beta_h m^h - \gamma_h n^h)\, (m_i\, n_j + m_j n_i) - (\gamma_h m^h - \delta_h n^h)\, n_i\, n_j$,

which by virtue of equation (4.3) can be expressed as

$$V_{ijkh} g^{ij} = \{2(L\, D_{(1)//h} + l_h\, D_{(1)}) - v_h\, (5\, D_{(3)} - D_{(2)})\}\, m_k + (L\, D_{//h} + l_h\, D)\, n_k \quad (4.8)$$

and

$$V_{ijkh} g^{kh} = \{L\, D_{(1)//h} m^h + L\, D_{(3)//h} n^h - 3(D_{(3)}\, v_{2)32} - D_{(1)}\, v_{2)33})\}\, m_i\, m_j + \{L\, D_{(1)//h} m^h + L\, D_{(2)//h} n^h - 3\, D_{(1)}\, v_{2)33} + (3D_{(2)} - 2D) v_{2)32}\}\, n_i\, n_j + \{L\, D_{(3)//h} m^h + L\, D_{(1)//h} n^h + 3\, D_{(1)}\, v_{2)32} + (3\, D_{(2)} - 2D) v_{2)33}\}.(m_I\, n_j + m_j n_i) \quad (4.9)$$

Hence:

Theorem 4.4

In a three-dimensional Finsler space F^3, V-tensor V_{ijkh} satisfies equations (4.8) and (4.9).

From equation (4.2), we can also obtain

$$V_{ijkh}\, l^i = 0,\ V_{ijkh} l^j = 0,\ V_{ijkh} l^k = 0, \quad (4.10)a$$

$$V_{ijkh} l^h = m_i\, m_j\, (\alpha_0 m_k + \beta_0 n_k) + (m_i\, n_j + m_j n_i)\, (\beta_0 m_k + \gamma_0 n_k) + n_i\, n_j \gamma_0 m_k, \quad (4.10)b$$

$$V_{ijkh} g^{ij} l^h = \beta_0 n_k,\ V_{ijkh} g^{ij} l^h m^k = 0 \quad ,\ V_{ijkh} g^{ij} n^k l^h = \beta_0. \quad (4.10)c$$

Hence:

Theorem 4.5

In a three-dimensional Finsler space F^3, tensor V_{ijkh} satisfies equations (4.10) a, b, c.

Equation (4.2) in a three-dimensional Finsler space F^3, can also be expressed as

$$V_{ijkh} = \sum_{(I,j,k)} \{A_{hk} h_{ij} + B_{hk} n_i\, n_j\} \quad (4.11)$$

where A_{hk} and B_{hk} are second order tensors defined by

$$A_{hk} = \{(1/3)\, \alpha_h m_k + \beta_h n_k\} \quad (4.12)$$

and

$$B_{hk} = [\{(1/3)\, \delta_h - \beta_h\}\, n_k - \{(1/3)\, \alpha_h + \gamma_h\}\, m_k] \quad (4.13)$$

From equations (4.12) and (4.13) we get

$$A_{hk} + B_{hk} = (1/3)\, \delta_h n_k - \gamma_h m_k \quad (4.14)a$$

$$A_{ok} = \{(1/3)\, \alpha_0 m_k + \beta_0 n_k\}, \quad (4.14)b$$

$B_{ok} = [\{(1/3)\ \delta_0 - \beta_0\}\ n_k - \{(1/3)\ \alpha_0 + \gamma_0\}\ m_k],$ (4.14)c

which imply

$A_{h0} + B_{h0} = 0$ (4.15)a

$A_{0k} + B_{0k} = (1/3)\ \delta_0 n_k - \gamma_0 m_k$ (4.15)b

Substituting the values of δ_0 and γ_0 from equation (4.7) we get

$A_{0k} + B_{0k} = (L\ D_{(1)//0} + D_{(1)})\ m_k + (1/3)(L\ D_{(2)//0} + D_{(2)})\ n_k$ (4.16)

Hence:

Theorem 4.6

In a three-dimensional Finsler space F^3, second order tensors A_{hk} and B_{hk} satisfy equation (4.14), and A_{0k} and B_{0k} satisfy equation (4.16).

In any three-dimensional Finsler space F^3, the V-tensor is defined by equation (4.11), which motivates us to give the following definition similar to T3-like Finsler space [8].

Definition 4.2

A Finsler space $F^n(n > 3)$, shall be called a V3-like Finsler space, if for arbitrary second order tensors A_{hk} and B_{hk}, satisfying $A_{h0} = 0$, $B_{h0} = 0$, its V-tensor, V_{ijkh} is non-zero and is expressed by an equation of the form (4.11).

Second Curvature Tensor

Let P_{hijk}, be the second curvature tensor in F^3 Matsumoto [6], then the Ricci tensors corresponding to them are defined as

$P_{hk}^{(1)} = P^j_{hjk} = C_{k/h} - C^j_{hk/j} + P^j_{kr}C^r_{jh} - P^r_{hk}\ C_r$ (5.1)

and

$P_{hk}^{(2)} = P^j_{hkj} = C_{k/h} - C^j_{hk/j} + C^r_{kh}\ C_{r/0} - P^j_{hr}C^r_{kj}$ (5.2)

These tensors are non-symmetric and satisfy

$P_{h0}^{(1)} = 0,\ P_{h0}^{(2)} = 0,\ P_{0k}^{(1)} = C_{k/0} = P_k,\ P_{0k}^{(2)} = C_{k/0} = P_k$ (5.3)

If we assume that the tensor $A_{hk} = P_{hk}^{(1)}$, we can obtain by virtue of equation (5.1) and $*V_{ij} = V_{ijkh}g^{kh}$, where $*V_{ij}$ is a symmetric tensor

$*V_{kh} = 4P_{hk}^{(1)} + B_{hk} + 2\ B_{hi}n^i n_k$ (5.4)

From equation (5.4) on simplification, we get

$Ç_{(h,k)}\{B_{ki}\ (\delta^i_h + 2\ n^i n_h) - 4\ (C_{k/h} + P^s_{kr}C^r_{sh})\} = 0,$ (4.5)

which leads to

$B_{k0} = B_{0i}\ (\delta^i_k + 2\ n^i n_k) + 4P_k$ (5.6)

From equation (5.6), we can easily obtain

$(B_{k0} – B_{0k})\, m^k = 4\, C_{/0}$, $(B_{k0} – 3\, B_{0k})\, n^k = 0$. (5.7)

Hence:

Theorem 5.1

In a D3-like, three-dimensional Finsler space F^3, if the tensor A_{hk} is given by $P_{hk}^{(!)}$, the tensor B_{k0} satisfies equations (5.6) and (5.7).

D-Reducible Finsler Space F^3

In a D-reducible Finsler space F^3, the tensor D_{ijk} satisfies Rastogi [9]

$$D_{ijk} = (1/4) \sum_{(I,j,k)} \{h_{ij}\, D_k\} \quad (6.1)$$

Equation (6.1) is similar to the one defined and studied by Matsumoto [4] for a C-reducible Finsler space. Equation (6.1) implies that $D_{(1)} = 0$, $D_{(2)} = (3/4)\, D$, $D_{(3)} = (1/4)\, D$, therefore from equation (2.3), we shall have

$$Q_{ijk} = (1/4)\, [D_{/0}\, (3\, n_i\, n_jn_k + m_i\, m_jn_k + m_jm_kn_i + m_k\, m_i\, n_j)\text{- } D\, h_0(3\, m_i\, m_jm_k + m_i\, n_jn_k + m_jn_kn_i + m_kn_i\, n_j)] \quad (6.2)$$

From equations (6.1) and (6.2), we can have

$$Q_{ijk} = D^{-1}D_{/0}\, D_{ijk} – (1/4)\, D\, h_0(3\, m_i\, m_jm_k + m_i\, n_jn_k+ m_jn_kn_i + m_kn_i\, n_j) \quad (6.3)$$

Following Izumi [2], we here give following definition:

Definition 6.1

A three-dimensional Finsler space F^3, for a constant λ, shall be called, a Q*- Finsler space, if the tensor $Q_{ijk} = \lambda\, D_{ijk}$.

From equation (6.3), we can observe that for a Q*-Finsler space F^3, $D^{-1}\, D_{/0} = \lambda$ and

$$D\, h_0\, (3\, m_i\, m_jm_k + m_i\, n_jn_k + m_jn_kn_i + m_kn_i\, n_j) = 0, \quad (6.4)$$

which on simplification implies $h_0 = 0$. Hence:

Theorem 6.1

In a D-reducible Q*-Finsler space F^3,$\lambda = (\log D)_{/0}$ and $h_0 = 0$.

From equations (6.1), we can obtain

$D_{(1)} = 0$, $D_{(2)} = (3/4)\, D$ and $D_{(3)} = (1/4)\, D$, ${}^1A_i = -(3/4)\, D\, h_i$, ${}^2A_i = (1/4)\, D\, h_i$,

${}^3A_i = (3/4)\, D_{/I}$, ${}^4A_i = (1/4)\, D_{/I}$, ${}^1A_0 = -(3/4)\, D\, h_0$, ${}^2A_0 = (1/4)\, D\, h_0$,

${}^3A_0 = (3/4)\, D_{/0}$, ${}^4A_0 = (1/4)\, D_{/0}$.

Using these values, we can obtain from equations (2.6) a, b, c, d

$${}^1A_{ij} = Ç_{(I,j)}\, \{(1/4)D_{/I}\, n_j – (3/4)\, D\, h_i\, m_j – (1/4)\, D^2\, h_0\, m_jn_i\}, \quad (6.5)a$$

$${}^2A_{ij} = Ç_{(I,j)}\, \{(1/4)\, D_{/I}\, m_j – (1/4)\, D\, h_i\, n_j – (1/8)\, D\, D_{/0}\, m_jn_i\}, \quad (6.5)b$$

$${}^3A_{ij} = Ç_{(I,j)}\, \{(1/4)\, D_{/I}\, m_j – (1/4)\, D\, h_i\, n_j + (1/8)\, D\, D_{/0}\, m_jn_i\}, \quad (6.5)c$$

$^4A_{ij} = Ç_{(I,j)} \{(3/4)\, D_{/I}\, n_j – (1/4)\, D\, h_i\, m_j – (1/8)\, D^2\, h_0\, m_j n_i\}$. (6.5)d

Substituting from equations (6.5) a, b, c, d in equation (2.4), on simplification we can obtain the value of U_{ijkh}, which easily gives

$$U_{ijkh} – U_{ijhk} = (1/4)\, D\, D_{/0}\, (m_j n_i – m_i\, n_j)\, (m_h n_k – m_k n_h) \quad (6.6)$$

From equation (6.6) we can obtain

Theorem 6.2

In a three-dimensional D-reducible Finsler space F^3, the symmetry of U_{ijkh} in k and h will imply $D_{/0} = 0$.

From equation (2.9) a, b for a D-reducible Finsler space F^3, we get

$$D'_{ijkh} = - (D^2/8)\, (h_{ik}h_{jh} – h_{ih}h_{jk}) \quad (6.7)\text{ a}$$

and

$$D'_{ik} = - (D^2/8)\, h_{ik} \quad (6.7)\text{b}$$

while from equation (2.10), we get

$$Ç_{(k,h)}\, [U_{ijkh} – (1/4)\, D\, D_{/0}\, (h_{ik}h_{jh} – h_{ih}h_{jk})] = 0 \quad (6.8)$$

From equations (6.7) a and (6.8) we can obtain

$$Ç_{(k,h)}\, [\, U_{ijkh} + 2\, D^{-1}D_{/0}D'_{ijkh}] = 0 \quad (6.9)$$

Hence:

Theorem 6.3

In a three-dimensional D-reducible Finsler space F^3, tensors U_{ijkh} and D'_{ijkh} are related by equation (6.9).

In case of a D-reducible Finsler space, from equations (3.6) and (3.7), on simplification we can obtain

$$Ç_{(I,h)}[U_{ih}^{(1)}– D_{/I}n_h + D\{h_i m_h + (3/16)\, (D_{/0} – D\, h_0)m_i n_h\}]= 0 \quad (6.10)$$

and

$$Ç_{(I,h)}[U_{ih}^{(2)} – D_{/I}n_h+ D\{h_i m_h – (3/8)\, D\, h_0\, m_i n_h\}] = 0 \quad (6.11)$$

Hence:

Theorem 6.4

In a three-dimensional D-reducible Finsler space F^3, tensors $U_{ih}^{(1)}$ and $U_{ih}^{(2)}$ satisfy equations (6.10) and (6.11).

In a three-dimensional D-reducible Finsler space F^3, from equations (4.3) a, b, c, d, we can obtain

$\alpha_h = - (3/4)\, D\, v_h$, $\beta_h = (1/4)\, (L\, D_{//h} + D\, l_h)$, $\gamma_h = - (1/4)\, D\, v_h$, $\delta_h = 3\, \beta_h$,

$\alpha_0 = 0$, $\beta_0 = (1/4)\, (L\, D_{//h}l^h + D)$, $\gamma_0 = 0$, $\delta_0 = 3\, \beta_0$,

$\alpha_h m^h = - (3/4)\, D\, v_{2)32}$, $\beta_h m^h = (1/4)\, (L\, D_{//h}m^h)$, $\gamma_h m^h = (1/3)\, \alpha_h m^h$,

$\delta_h m^h = 3\,\beta_h m^h$, $\alpha_h n^h = -\,(3/4)\,D\,v_{2)33}$, $\beta_h n^h = (1/4)(L\,D_{//h} n^h)$,

$\gamma_h n^h = (1/3)\,\alpha_h n^h$, $\delta_h n^h = 3\,\beta_h n^h$.

From equations (4.8), (4.9) and (4.10) b, for a D-reducible Finsler space F^3, we get

$$V_{ijkh} g^{ij} = (L\,D_{//h} + D\,l_h)\,n_k - (3/2)\,D\,m_k, \tag{6.12)a}$$

$$V_{ijkh} g^{kh} = (1/4)\,[(L\,D_{//h} n^h - 3\,D\,v_{2)32})\,m_i\,m_j + (3\,L\,D_{//h} n^h + D\,v_{2)32})\,n_i\,n_j$$

$$+ (L\,D_{//h} m^h + D\,v_{2)33})\,(m_i\,m_j + n_i\,n_j), \tag{6.12)b}$$

$$V_{ijkkh} l^h = (1/4)(L\,D_{//0} + D)\sum_{(I,j,k)} \{m_i\,m_j n_k\} \tag{6.12)c}$$

Hence:

Theorem 6.5

In a three-dimensional D-reducible Finsler space F^3, tensor V_{ijkh} satisfies equations (6.12) a, b, c.

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